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AROUND GENERIC LINEAR PERTURBATIONS

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1. INTRODUCTION

In this paper, ℓ , m and n stand for positive integers. Throughout this paper, unless otherwise stated, all manifolds and mappings belong to class C^∞ and all manifolds are without boundary. The purpose of this paper is to introduce some results shown in [2, 3].

Let $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^\ell$, U and $F : U \rightarrow \mathbb{R}^\ell$ be a linear mapping, an open subset of \mathbb{R}^m and a mapping, respectively.

Set

$$F_\pi = F + \pi.$$

Here, π in $F_\pi = F + \pi$ is restricted to the open set U .

Let $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the space consisting of all linear mappings of \mathbb{R}^m into \mathbb{R}^ℓ . Notice that we have the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$. An n -dimensional manifold is denoted by N .

In Section 2, two main theorems of [2] (Theorems 1 and 2) are introduced. Theorem 1 is as follows. Let $f : N \rightarrow U$ (resp., $F : U \rightarrow \mathbb{R}^\ell$) be an immersion (resp., a mapping). Generally, the composition $F \circ f$ does not necessarily yield a mapping which is transverse to a given subfiber-bundle of the jet bundle $J^1(N, \mathbb{R}^\ell)$. Nevertheless, Theorem 1 asserts that for any \mathcal{A}^1 -invariant fiber, a generic mapping $F_\pi \circ f$ yields a mapping which is transverse to the subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the given fiber. Theorem 2 is a specialized transversality result on crossings of a generic mapping $F_\pi \circ f$, where $f : N \rightarrow U$ (resp., $F : U \rightarrow \mathbb{R}^\ell$) is a given injection (resp., a given mapping).

In Section 3, some applications of Theorems 1 and 2 are introduced.

In Section 4, the main result of [3] (Theorem 4) is introduced. Theorem 4 is as follows. In [4], John Mather proved that almost all linear projections from a submanifold of a vector space into a subspace are transverse with respect to a given modular submanifold. Theorem 4 is an improvement of the result. Namely, almost all linear perturbations of a smooth mapping from a submanifold of \mathbb{R}^m into \mathbb{R}^ℓ yield a transverse mapping with respect to a given modular submanifold.

2. COMPOSING GENERIC LINEARLY PERTURBED MAPPINGS AND IMMERSIONS/INJECTIONS

In the following, we denote manifolds by N and P .

Definition 1. Let W be a submanifold of P , and let $g : N \rightarrow P$ be a mapping.

- (1) We say that $g : N \rightarrow P$ is *transverse* to W at q if $g(q) \notin W$ or in the case of $g(q) \in W$, the following holds:

$$dg_q(T_q N) + T_{g(q)} W = T_{g(q)} P.$$

- (2) We say that $g : N \rightarrow P$ is *transverse* to W if for any $q \in N$, g is transverse to W at q .

We say that $g : N \rightarrow P$ is \mathcal{A} -*equivalent* to $h : N \rightarrow P$ if there exist two diffeomorphisms $\Phi : N \rightarrow N$ and $\Psi : P \rightarrow P$ such that $g = \Psi \circ h \circ \Phi^{-1}$.

Let $J^r(N, P)$ denote the space of r -jets of mappings of N into P . For a given mapping $g : N \rightarrow P$, the mapping $j^r g : N \rightarrow J^r(N, P)$ is given by $q \mapsto j^r g(q)$ (for details on $J^r(N, P)$ or $j^r g : N \rightarrow J^r(N, P)$, see for instance, [1]).

In order to state Theorem 1, it is sufficient to consider the case of $r = 1$ and $P = \mathbb{R}^\ell$. Let $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ denote a coordinate neighborhood system of N . Let $\Pi : J^1(N, \mathbb{R}^\ell) \rightarrow N \times \mathbb{R}^\ell$ denote the natural projection defined by $\Pi(j^1 g(q)) = (q, g(q))$. Let $\Phi_\lambda : \Pi^{-1}(U_\lambda \times \mathbb{R}^\ell) \rightarrow \varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times J^1(n, \ell)$ denote the homeomorphism given by

$$\Phi_\lambda(j^1 g(q)) = (\varphi_\lambda(q), g(q), j^1(\psi_\lambda \circ g \circ \varphi_\lambda^{-1} \circ \tilde{\varphi}_\lambda)(0)),$$

where $J^1(n, \ell) = \{j^1 g(0) \mid g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^\ell, 0)\}$ and $\tilde{\varphi}_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (resp., $\psi_\lambda : \mathbb{R}^m \rightarrow \mathbb{R}^m$) is the translation defined by $\tilde{\varphi}_\lambda(0) = \varphi_\lambda(q)$ (resp., $\psi_\lambda(g(q)) = 0$). Then, $\{(\Pi^{-1}(U_\lambda \times \mathbb{R}^\ell), \Phi_\lambda)\}_{\lambda \in \Lambda}$ is a coordinate neighborhood system of $J^1(N, \mathbb{R}^\ell)$. We say that a subset $X \subset J^1(n, \ell)$ is \mathcal{A}^1 -*invariant* if for any $j^1 g(0) \in X$, and for any two germs of diffeomorphisms $H : (\mathbb{R}^\ell, 0) \rightarrow (\mathbb{R}^\ell, 0)$ and $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$, we get $j^1(H \circ g \circ h^{-1})(0) \in X$. Let X denote an \mathcal{A}^1 -invariant submanifold of $J^1(n, \ell)$. Set

$$X(N, \mathbb{R}^\ell) = \bigcup_{\lambda \in \Lambda} \Phi_\lambda^{-1}(\varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times X).$$

Then, $X(N, \mathbb{R}^\ell)$ is a subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the fiber X satisfying

$$\begin{aligned} \text{codim } X(N, \mathbb{R}^\ell) &= \dim J^1(N, \mathbb{R}^\ell) - \dim X(N, \mathbb{R}^\ell) \\ &= \dim J^1(n, \ell) - \dim X \\ &= \text{codim } X. \end{aligned}$$

Theorem 1 ([2]). *Let f be an immersion of N into an open subset U of \mathbb{R}^m , where N is a manifold of dimension n . Let $F : U \rightarrow \mathbb{R}^\ell$ be a mapping. If X is an \mathcal{A}^1 -invariant submanifold of $J^1(n, \ell)$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $j^1(F_\pi \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$ is transverse to $X(N, \mathbb{R}^\ell)$.*

Now, for the statement of Theorem 2, we will prepare some definitions. Set $N^{(s)} = \{(q_1, q_2, \dots, q_s) \in N^s \mid q_i \neq q_j \text{ (} i \neq j \text{)}\}$. Note that $N^{(s)}$ is an open submanifold of N^s . For a given mapping $g : N \rightarrow P$, let $g^{(s)} : N^{(s)} \rightarrow P^s$ be the mapping defined by

$$g^{(s)}(q_1, q_2, \dots, q_s) = (g(q_1), g(q_2), \dots, g(q_s)).$$

Set $\Delta_s = \{(y, \dots, y) \in P^s \mid y \in P\}$. It is not hard to see that Δ_s is a submanifold of P^s satisfying

$$\text{codim } \Delta_s = \dim P^s - \dim \Delta_s = (s-1)\dim P.$$

Definition 2. We say that $g : N \rightarrow P$ is a *mapping with normal crossings* if for any positive integer s ($s \geq 2$), $g^{(s)} : N^{(s)} \rightarrow P^s$ is transverse to Δ_s .

For any injection $f : N \rightarrow \mathbb{R}^m$, set

$$s_f = \max \left\{ s \mid \forall (q_1, q_2, \dots, q_s) \in N^{(s)}, \dim \sum_{i=2}^s \overline{\mathbb{R}f(q_1)f(q_i)} = s - 1 \right\}.$$

Since the mapping f is injective, it follows that $2 \leq s_f$. Since $f(q_1), f(q_2), \dots, f(q_{s_f})$ are points of \mathbb{R}^m , we have $s_f \leq m + 1$. Hence, we get

$$2 \leq s_f \leq m + 1.$$

Moreover, in the following, for a set X , we denote the number of its elements (or its cardinality) by $|X|$.

Theorem 2 ([2]). *Let f be an injection of N into an open subset U of \mathbb{R}^m , where N is a manifold of dimension n . Let $F : U \rightarrow \mathbb{R}^\ell$ be a mapping. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any s ($2 \leq s \leq s_f$), $(F_\pi \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s$ is transverse to Δ_s . Furthermore, if the mapping F_π satisfies that $|F_\pi^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^\ell$, then $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is a mapping with normal crossings.*

3. APPLICATIONS OF THEOREMS 1 AND 2

In Subsection 3.1 (resp., Subsection 3.2), applications of Theorem 1 (resp., Theorem 2) are stated.

3.1. Applications of Theorem 1. Set

$$\Sigma^k = \{j^1 g(0) \in J^1(n, \ell) \mid \text{corank } Jg(0) = k\},$$

where $\text{corank } Jg(0) = \min\{n, \ell\} - \text{rank } Jg(0)$ and $k = 1, 2, \dots, \min\{n, \ell\}$. Then, Σ^k is an \mathcal{A}^1 -invariant submanifold of $J^1(n, \ell)$. Set

$$\Sigma^k(N, \mathbb{R}^\ell) = \bigcup_{\lambda \in \Lambda} \Phi_\lambda^{-1}(\varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times \Sigma^k),$$

where Φ_λ and φ_λ are as defined in Section 2. Then, the set $\Sigma^k(N, \mathbb{R}^\ell)$ is a subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the fiber Σ^k satisfying

$$\begin{aligned} \text{codim } \Sigma^k(N, \mathbb{R}^\ell) &= \dim J^1(N, \mathbb{R}^\ell) - \dim \Sigma^k(N, \mathbb{R}^\ell) \\ &= (n - v + k)(\ell - v + k), \end{aligned}$$

where $v = \min\{n, \ell\}$. (For details on Σ^k and $\Sigma^k(N, \mathbb{R}^\ell)$, see for instance [1], pp. 60–61).

As applications of Theorem 1, we get the following Proposition 1, Corollaries 1, 2, 3 and 4.

Proposition 1 ([2]). *Let f be an immersion of N into an open subset U of \mathbb{R}^m , where N is a manifold of dimension n . Let $F : U \rightarrow \mathbb{R}^\ell$ be a mapping. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \rightarrow J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$ for any positive integer k satisfying $1 \leq k \leq v$. Especially, in the case of $\ell \geq 2$, we get $k_0 + 1 \leq v$ and it follows that $j^1(F_\pi \circ f)$ satisfies that $j^1(F_\pi \circ f)(N) \cap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ for any positive integer k ($k_0 + 1 \leq k \leq v$), where k_0 is the maximum integer satisfying $(n - v + k_0)(\ell - v + k_0) \leq n$ ($v = \min\{n, \ell\}$).*

- Remark 1.** (1) In Proposition 1, by $(n - v + k_0)(\ell - v + k_0) \leq n$, it is not hard to see that $k_0 \geq 0$.
- (2) In Proposition 1, in the case of $\ell = 1$, we get $k_0 + 1 > v$. Indeed, in the case, by $v = 1$, we have $(n - 1 + k_0)k_0 \leq n$. Thus, it follows that $k_0 = 1$.

A mapping $g : N \rightarrow \mathbb{R}$ is called a *Morse function* if all of the singularities of g are nondegenerate (for details on Morse functions, see for instance, [1], p. 63). In the case of $(n, \ell) = (n, 1)$, we get the following.

Corollary 1 ([2]). *Let f be an immersion of N into an open subset U of \mathbb{R}^m , where N is a manifold of dimension n . Let $F : U \rightarrow \mathbb{R}$ be a mapping. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$, the mapping $F_\pi \circ f : N \rightarrow \mathbb{R}$ is a Morse function.*

For a given mapping $g : N \rightarrow \mathbb{R}^{2n-1}$ ($n \geq 2$), a singular point $q \in N$ is called a *singular point of Whitney umbrella* if there exist two germs of diffeomorphisms $H : (\mathbb{R}^{2n-1}, g(q)) \rightarrow (\mathbb{R}^{2n-1}, 0)$ and $h : (N, q) \rightarrow (\mathbb{R}^n, 0)$ satisfying $H \circ g \circ h^{-1}(x_1, x_2, \dots, x_n) = (x_1^2, x_1x_2, \dots, x_1x_n, x_2, \dots, x_n)$, where (x_1, x_2, \dots, x_n) is a local coordinate around the point $h(q) = 0 \in \mathbb{R}^n$. In the case of $(n, \ell) = (n, 2n - 1)$ ($n \geq 2$), we get the following.

Corollary 2 ([2]). *Let f be an immersion of N into an open subset U of \mathbb{R}^m , where N is a manifold of dimension n ($n \geq 2$). Let $F : U \rightarrow \mathbb{R}^{2n-1}$ be a mapping. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1})$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma$, any singular point of the mapping $F_\pi \circ f : N \rightarrow \mathbb{R}^{2n-1}$ is a singular point of Whitney umbrella.*

In the case of $\ell \geq 2n$, the immersion property of a given mapping $f : N \rightarrow U$ is preserved by composing generic linearly perturbed mappings as follows:

Corollary 3 ([2]). *Let f be an immersion of N into an open subset U of \mathbb{R}^m , where N is a manifold of dimension n . Let $F : U \rightarrow \mathbb{R}^\ell$ be a mapping ($\ell \geq 2n$). Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is an immersion.*

A mapping $g : N \rightarrow \mathbb{R}^\ell$ has corank at most k singular points if

$$\sup \{ \text{corank } dg_q \mid q \in N \} \leq k,$$

where $\text{corank } dg_q = \min\{n, \ell\} - \text{rank } dg_q$. From Proposition 1, we have the following.

Corollary 4 ([2]). *Let f be an immersion of N into an open subset U of \mathbb{R}^m , where N is a manifold of dimension n . Let $F : U \rightarrow \mathbb{R}^\ell$ be a mapping. Let k_0 be the maximum integer satisfying $(n - v + k_0)(\ell - v + k_0) \leq n$ ($v = \min\{n, \ell\}$). Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ has corank at most k_0 singular points.*

3.2. Applications of Theorem 2.

Proposition 2 ([2]). *Let f be an injection of N into an open subset U of \mathbb{R}^m , where N is a manifold of dimension n . Let $F : U \rightarrow \mathbb{R}^\ell$ be a mapping. If $(s_f - 1)\ell > ns_f$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is a mapping with normal crossings satisfying $(F_\pi \circ f)^{(s_f)}(N^{(s_f)}) \cap \Delta_{s_f} = \emptyset$.*

In the case of $\ell > 2n$, the injection property of a given mapping $f : N \rightarrow U$ is preserved by composing generic linearly perturbed mappings as follows:

Corollary 5 ([2]). *Let f be an injection of N into an open subset U of \mathbb{R}^m , where N is a manifold of dimension n . Let $F : U \rightarrow \mathbb{R}^\ell$ be a mapping. If $\ell > 2n$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is injective.*

By combining Corollaries 3 and 5, we get the following.

Corollary 6 ([2]). *Let f be an injective immersion of N into an open subset U of \mathbb{R}^m , where N is a manifold of dimension n . Let $F : U \rightarrow \mathbb{R}^\ell$ be a mapping. If $\ell > 2n$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is an injective immersion.*

In Corollary 6, suppose that the mapping $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is proper. Then, an injective immersion $F_\pi \circ f$ is necessarily an embedding (see [1], p.11). Hence, we have the following.

Corollary 7 ([2]). *Let f be an embedding of N into an open subset U of \mathbb{R}^m , where N is a compact manifold of dimension n . Let $F : U \rightarrow \mathbb{R}^\ell$ be a mapping. If $\ell > 2n$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is an embedding.*

4. COMPOSING GENERIC LINEARLY PERTURBED MAPPINGS AND EMBEDDINGS

Let $C^\infty(N, P)$ be the set consisting of all C^∞ mappings of N into P , and the topology on $C^\infty(N, P)$ is the Whitney C^∞ topology (for the definition of Whitney C^∞ topology, see for instance [1]). Then, we say that g is *stable* if the \mathcal{A} -equivalence class of g is open in $C^\infty(N, P)$.

Let ${}_s J^r(N, P)$ be the space consisting of elements $(j^r g(q_1), \dots, j^r g(q_s)) \in J^r(N, P)^s$ satisfying $(q_1, \dots, q_s) \in N^{(s)}$, where s is a positive integer. Since $N^{(s)}$ is an open submanifold of N^s , the space ${}_s J^r(N, P)$ is also an open submanifold of $J^r(N, P)^s$. For a given mapping $g : N \rightarrow P$, ${}_s j^r g : N^{(s)} \rightarrow {}_s J^r(N, P)$ is defined by $(q_1, \dots, q_s) \mapsto (j^r g(q_1), \dots, j^r g(q_s))$.

Let W be a submanifold of ${}_s J^r(N, P)$. We say that a mapping $g : N \rightarrow P$ is *transverse with respect to W* if ${}_s j^r g : N^{(s)} \rightarrow {}_s J^r(N, P)$ is transverse to W .

Following Mather ([4]), we can partition P^s as follows. For any partition π of $\{1, \dots, s\}$, let P^π be the set of s -tuples $(y_1, \dots, y_s) \in P^s$ such that $y_i = y_j$ if and only if two positive integers i and j are in the same member of the partition π .

Let $\text{Diff } N$ be the group of diffeomorphisms of N . We have a natural action of $\text{Diff } N \times \text{Diff } P$ on ${}_s J^r(N, P)$ such that for a mapping $g : N \rightarrow P$, the equality $(h, H) \cdot {}_s j^r g(q) = {}_s j^r (H \circ g \circ h^{-1})(q')$ holds, where $q = (q_1, \dots, q_s)$ and $q' = (h(q_1), \dots, h(q_s))$. We say that a subset $W \subset {}_s J^r(N, P)$ is *invariant* if it is invariant under this action.

We recall the following identification $(*)$ from [4]. Let $q = (q_1, \dots, q_s) \in N^{(s)}$, let $g : U \rightarrow P$ be a mapping defined in a neighborhood U of $\{q_1, \dots, q_s\}$ in N , and let $z = {}_s j^r g(q)$, $q' = (g(q_1), \dots, g(q_s))$. Let ${}_s J^r(N, P)_q$ and ${}_s J^r(N, P)_{q, q'}$ be the fibers of ${}_s J^r(N, P)$ over q and over (q, q') respectively. Let $J^r(N)_q$ be the \mathbb{R} -algebra of r -jets at q of functions on N . Namely, we have

$$J^r(N)_q = {}_s J^r(N, \mathbb{R})_q.$$

Set $g^*TP = \bigcup_{\tilde{q} \in U} T_{g(\tilde{q})}P$, where TP is the tangent bundle of P . Let $J^r(g^*TP)_q$ denote the $J^r(N)_q$ -module of r -jets at q of sections of the bundle g^*TP . Let \mathfrak{m}_q be the ideal in $J^r(N)_q$ consisting of jets of functions which vanish at the point q . Namely, we have

$$\mathfrak{m}_q = \{ {}_s j^r h(q) \in {}_s J^r(N, \mathbb{R})_q \mid h(q_1) = \cdots = h(q_s) = 0 \}.$$

Let $\mathfrak{m}_q J^r(g^*TP)_q$ denote the set consisting of finite sums of products of an element of \mathfrak{m}_q and an element of $J^r(g^*TP)_q$. Namely, we have

$$\mathfrak{m}_q J^r(g^*TP)_q = J^r(g^*TP)_q \cap \{ {}_s j^r \xi(q) \in {}_s J^r(N, TP)_q \mid \xi(q_1) = \cdots = \xi(q_s) = 0 \}.$$

Then, the following canonical identification of \mathbb{R} vector spaces $(*)$ holds.

$$(*) \quad T({}_s J^r(N, P)_{q, q'})_z = \mathfrak{m}_q J^r(g^*TP)_q.$$

Now, let W be a non-empty submanifold of ${}_s J^r(N, P)$. Choose $q = (q_1, \dots, q_s) \in N^{(s)}$ and $g : N \rightarrow P$, and let $z = {}_s j^r g(q)$ and $q' = (g(q_1), \dots, g(q_s))$. Suppose that $z \in W$. Set $W_{q, q'} = \tilde{\pi}^{-1}(q, q')$, where $\tilde{\pi} : W \rightarrow N^{(s)} \times P^s$ is defined by $\tilde{\pi}({}_s j^r \tilde{g}(\tilde{q})) = (\tilde{q}, (\tilde{g}(\tilde{q}_1), \dots, \tilde{g}(\tilde{q}_s)))$ and $\tilde{q} = (\tilde{q}_1, \dots, \tilde{q}_s) \in N^{(s)}$. Suppose that $W_{q, q'}$ is a submanifold of ${}_s J^r(N, P)$. Then, from $(*)$, the tangent space $T(W_{q, q'})_z$ can be identified with a vector subspace of $\mathfrak{m}_q J^r(g^*TP)_q$. By $E(g, q, W)$, we denote this vector subspace.

Definition 3. A submanifold W of ${}_s J^r(N, P)$ is said to be *modular* if conditions (α) and (β) below are satisfied:

- (α) The set W is an invariant submanifold of ${}_s J^r(N, P)$, and lies over P^π for some partition π of $\{1, \dots, s\}$.
- (β) For any $q \in N^{(s)}$ and any mapping $g : N \rightarrow P$ satisfying ${}_s j^r g(q) \in W$, the subspace $E(g, q, W)$ is a $J^r(N)_q$ -submodule.

Now, suppose that $P = \mathbb{R}^\ell$. The main theorem of [4] is the following.

Theorem 3 ([4]). *Let f be an embedding of N into \mathbb{R}^m , where N is a manifold of dimension n . If W is a modular submanifold of ${}_s J^r(N, \mathbb{R}^\ell)$ and $m > \ell$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is transverse with respect to W .*

Theorem 4 ([3]). *Let f be an embedding of N into an open subset U of \mathbb{R}^m , where N is a manifold of dimension n . Let $F : U \rightarrow \mathbb{R}^\ell$ be a mapping. If W is a modular submanifold of ${}_s J^r(N, \mathbb{R}^\ell)$, then there exists a subset Σ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is transverse with respect to W .*

By the same way as in the proof of Theorem 3 of [4], we get the following as a corollary of Theorem 4.

Corollary 8 ([3]). *Let f be an embedding of N into an open subset U of \mathbb{R}^m , where N is a compact manifold of dimension n . Let $F : U \rightarrow \mathbb{R}^\ell$ be a mapping. If a dimension pair (n, ℓ) is in the nice dimensions, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the composition $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is stable.*

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REFERENCES

- [1] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics 14, Springer, New York, 1973.
- [2] S. Ichiki, *Composing generic linearly perturbed mappings and immersions/injections*, to appear in J. Math. Soc. Japan, available from arXiv:1612.01100.
- [3] S. Ichiki, *Generic linear perturbations*, to appear in Proc. Amer. Math. Soc., available from arXiv:1607.03220.
- [4] J. N. Mather, *Generic projections*, Ann. of Math., (2) **98** (1973), 226–245.

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